

The eternal fractal in the universe

Serge Winitzki

Department of Physics and Astronomy,

Tufts University, Medford, MA 02155, USA

Models of eternal inflation predict a stochastic self-similar geometry of the universe at very large scales and allow existence of points that never thermalize. I explore the fractal geometry of the resulting spacetime, using coordinate-independent quantities. The formalism of stochastic inflation can be used to obtain the fractal dimension of the set of eternally inflating points (the “eternal fractal”). I also derive a nonlinear branching diffusion equation describing global properties of the eternal set and the probability to realize eternal inflation. I show gauge invariance of the condition for presence of eternal inflation. Finally, I consider the question of whether all thermalized regions merge into one connected domain. Fractal dimension of the eternal set provides a (weak) sufficient condition for merging.

PACS numbers: 98.80.Hw, 98.80.Bp

I. INTRODUCTION

In models of cosmological inflation, amplification of quantum fluctuations of the scalar field (the inflaton) may result in eternal self-reproduction of inflating domains, or *eternal inflation* [1, 2, 3, 4]. In this case, stochastic evolution produces a spacetime in which thermalization never happens globally. At arbitrarily late times there exist vast regions of space where inflation still continues; however, along any given comoving world-line, thermalization is certain to be reached (with probability 1) at a sufficiently late proper time. This somewhat paradoxical situation is possible when self-reproduction of inflating regions proceeds faster than the drift of the inflaton field toward thermalization within these regions.

To study the spacetime resulting from eternal inflation, one may use a stochastic formalism [1, 5, 6, 7] that describes the probability distribution of the inflaton field ϕ , coarse-grained in horizon-size volumes of space. In simplest one-field models of inflation, the time-dependent

distribution of physical volume $P_V(\phi, t)$ satisfies a Fokker-Planck (FP) equation

$$\frac{\partial P_V}{\partial t} = \frac{\partial^2}{\partial \phi^2} (D P_V) - \frac{\partial}{\partial \phi} (v P_V) + 3H P_V. \quad (1)$$

This equation¹ is supplemented by appropriate initial and boundary conditions; $D(\phi)$ and $v(\phi)$ are diffusion and drift coefficients and $H(\phi)$ is the effective expansion rate. The kinetic coefficients are related to the inflaton potential $V(\phi)$ by

$$H(\phi) = \sqrt{\frac{8\pi V(\phi)}{3}}, \quad D = \frac{H^3}{8\pi^2}, \quad v = -\frac{1}{4\pi} \frac{dH}{d\phi}. \quad (2)$$

At late times, solutions of Eq. (1) are dominated by the eigenfunction $\psi_V(\phi)$ of the diffusion operator with the largest eigenvalue γ_V ,

$$P_V(\phi, t) \sim e^{\gamma_V t} \psi_V(\phi). \quad (3)$$

If $\gamma_V > 0$, the total volume of the inflating domain $\int P_V(\phi, t) d\phi$ grows exponentially with time, which indicates presence of eternal inflation.

One problem with the present form of the stochastic formalism is its dependence on the choice of equal-time surfaces. By construction, the volume distribution at constant time $P_V(\phi, t)$ is not a generally covariant quantity, and it is no surprise that solutions and eigenvalues of the FP equation depend non-trivially on the choice of the time variable t . The resulting spacetime is highly inhomogeneous on large scales, and different choices of time slicing introduce significant biases into the volume distribution. For this reason, it has been difficult to draw unambiguous conclusions about eternally inflating spacetimes using stochastic formalism [3, 7, 9]. One could even question the meaning of the condition for presence of eternal inflation, $\gamma_V > 0$, since the eigenvalue γ_V is itself gauge-dependent.

Eternal self-reproduction of inflating regions gives rise to fractal structure of eternally inflating spacetime on very large scales. Previously, the fractal structure of the inflating domain has been investigated using distributions of ϕ on equal-time surfaces [7, 10, 11]. However, the fractal dimension d_{fr} of the inflating domain obtained in this way is determined by an eigenvalue of the FP equation and depends, in general, on the choice of time parametrization (except for the case when $H(\phi) \equiv \text{const}$).

Another open issue concerns the global topology of the eternally self-reproducing universe: do thermalized domains remain forever separated from each other by an inflating sea, or do

¹ Here and below we use the Ito factor ordering of the diffusion operator [8].

all thermalized domains eventually merge into one connected domain surrounding inflating islands? If merging occurs, all other thermalized domains in the universe will eventually enter our horizon; otherwise, we shall remain forever causally separated from other thermalized domains by eternally inflating walls. An early work by Guth and Weinberg [12] showed that in the old inflationary scenario there is no percolation of thermalized bubbles. The same occurs in certain models of topological inflation [13] with inflating domain walls. Numerical simulations of Linde *et al.* [7, 14] suggest that thermalized regions merge in some chaotic inflationary scenarios. However, a conclusive general description of the topology of the thermalized domain is lacking.

In the present work, I consider models of inflation that can be described by Eq. (1) or its generalizations for multiple scalar fields. I demonstrate that certain global properties of eternally inflating spacetimes can be determined in a coordinate-independent way. The principal object under consideration is the “eternal fractal set” E , the set of all comoving points that never thermalize. The set E is defined (Sec. II) independently of the choice of coordinates. I show (Sec. II D) that its fractal dimension, $\dim E$, is a gauge-invariant quantity which coincides with the fractal dimension d_{fr} of the inflating domain computed using the scale factor time variable. In Sec. III I study conditions for presence of eternal inflation. First, I show that the basic criterion, $\gamma_V > 0$, is in fact gauge-invariant. Then I define the probability $X(\phi)$ to have eternal points in a given comoving domain; this probability plays the role of density of points of E . The quantity $X(\phi)$ is gauge-independent and satisfies a nonlinear branching diffusion equation derived in Sec. III B. This new equation is shown (Sec. III D) to provide a (sufficient) condition for presence of eternal inflation that is consistent with the first criterion $\gamma_V > 0$. Finally, in Sec. IV I consider the question of whether all thermalized domains ultimately merge and become connected to each other at sufficiently late times. I show that the probability for merging is constrained by the nonlinear diffusion equation of Sec. III B. A (weak) sufficient condition for merging, $\dim E < 2$, is provided by the fractal dimension of the set E . I conjecture that the fractal dimension may provide a stronger criterion of merging of thermalized domains. I also generalize the results to a suitable class of global topological properties of the set E .

II. THE ETERNAL FRACTAL

In the current picture of eternal inflation, certain regions of spacetime may expand by an arbitrary factor and, in particular, arbitrarily small sub-Planckian scales may become macroscopically large. There is evidence that spacetime cannot be considered classical below the Planck scale (see Ref. [15] for a review). Recently, it was shown that a hypothetical modification of physics on sub-Planckian scales may change the power spectrum of inflationary perturbations [16] and create a significant backreaction affecting the evolution [17]. In that case, the kinetic coefficients of Eq. (1) would have to be modified. However, the qualitative features of an eternally self-reproducing spacetime would remain the same. In the present work we shall assume that no new physics emerges from previously sub-Planckian scales in the course of inflationary self-reproduction.

Let us consider the evolution of an arbitrary finite (comoving) volume of space chosen at some initial time, in a cosmological model with eternal inflation. Here and below we assume that the initial comoving volume at $t = 0$ had a certain value ϕ_0 of the inflaton field and was physically of one horizon size $H^{-1}(\phi_0)$.

There is a non-vanishing, if small, probability that the entire comoving region will thermalize at a certain finite time, because fluctuations of the scalar field ϕ could have accidentally cooperated to drive ϕ toward the end point of inflation ϕ_* everywhere in the region. One could say in that case that eternal inflation has not been realized in that region.² If, on the other hand, eternal inflation has been realized, then for an arbitrarily late time t there will be regions that are still inflating at that time.

In that case, there must exist some comoving world-lines that never enter a thermalized domain. To demonstrate this, one could choose a monotonically increasing sequence of time instances, $t_n = nH^{-1}$, $n = 0, 1, 2, \dots$; for each n there must exist a point x_n surrounded by a horizon-sized domain that will be still inflating at time t_n . (As t_n grows, these domains become progressively smaller in comoving coordinates, due to expansion of space.) An infinite sequence of points x_n on a finite (compact) comoving volume of 3-dimensional space must have accumulation points, *i.e.* there must be at least one point x_* such that any arbitrarily small (comoving) neighborhood of x_* contains infinitely many points x_n from

² Here we do not consider the possibility of spontaneous return of thermalized regions to inflation in “recycling universe” scenarios [18].

the sequence. It is clear that the comoving world-line at an accumulation point x_* cannot reach thermalization at any finite time: if it did, there would exist a comoving neighborhood around x_* which thermalized at that time, and this contradicts the construction of the point x_* . We shall refer to points x_* that never thermalize as “eternal points” and define the set E of all such points, as a set drawn on the spatial section of the comoving volume at initial time. (One could also imagine an infinitely dense grid of comoving world-lines starting at the initial surface, with the set E consisting of all world-lines that never reach thermalization.) The set E for a comoving region is not empty if eternal inflation has been realized in that region.

Different histories and different choices of the initial comoving volume will generate different sets E . Since E is a stochastically generated set, one may characterize it probabilistically, by finding probabilities for the set E to have certain properties. The underlying stochastic process is the random walk of the inflaton ϕ and, since it is a stationary stochastic process that does not explicitly depend on time, probabilities of any properties of the set E depend only on ϕ_0 . For instance, below we shall denote by $X(\phi_0)$ the probability for the set E to be non-empty if the initial comoving volume has horizon size and has a given value $\phi = \phi_0$ of the inflaton field.

During stochastic evolution, there will be (infinitely many) times when the inflaton returns to the value ϕ_0 in some horizon-sized domain; each time the distribution of properties of the subset of E within those domains will be the same as that for the whole set E . In this way, the set E naturally acquires fractal structure.

A. The fractal dimension

The comoving volume $P(\phi, t)$ of inflating regions with field value ϕ at time t is described by the FP equation

$$\frac{\partial P}{\partial t} = \frac{\partial^2}{\partial \phi^2} (D(\phi) P) - \frac{\partial}{\partial \phi} (v(\phi) P) \quad (4)$$

with the kinetic coefficients from Eq. (2). Its late-time asymptotic solution is

$$P(\phi, t) \sim e^{-\gamma t} \psi(\phi). \quad (5)$$

The total comoving volume of inflating regions $\int P(\phi, t) d\phi$ decays exponentially rapidly at late times, and any pre-selected comoving worldline eventually reaches thermalization with

probability 1. In other words, the comoving volume occupied by eternal points vanishes and the eternal set E has measure zero, as a subset of 3-dimensional (comoving) space. Its fractal dimension must therefore be less than three.

The fractal set E can be thought of as an infinite-time limit of the (fractal) inflating domain considered in Refs. [7, 10, 11]. Since the volume of E vanishes, we need a slightly different definition of fractal dimension than that of Ref. [10]. The definition of fractal dimension we would like to use is the so-called “box fractal dimension”, which in our case is equivalent to the Hausdorff-Besicovitch fractal dimension [20]. The “box fractal dimension” is defined for a non-empty subset S of a finite region of a d -dimensional space. Rectangular coordinates are chosen in the region to divide it into a regular lattice of d -dimensional cubes with side length l . (The cubical shape of the lattice is not essential to the definition of the “box fractal dimension”.) Then, the “coarse-grained volume” $V(l)$ of the set S is defined as the total volume of all cubes of the lattice that contain at least one point from the set S . If the set S has fractal dimension less than d , one expects $V(l)$ to decay at $l \rightarrow 0$ as some power of l . The fractal dimension is defined by

$$\dim S = d - \lim_{l \rightarrow 0} \frac{\ln V(l)}{\ln l}. \quad (6)$$

For example, if the set S is made of a finite number of line segments, then the coarse-grained volume at scale l will be equal to the volume of narrow tubes of width l surrounding the line segments. Then $V(l) \sim l^{d-1}$ and the fractal dimension defined by Eq. (6) would come out to be 1.

Below we shall see that if the eternal set E is not empty, it has a well-defined fractal dimension $\dim E$ which is independent of the choice of spacetime coordinates, and we shall give a procedure to compute $\dim E$ in a given inflationary model.

B. Topology of the eternal fractal

Some simple topological properties of the eternal set E follow directly from its construction.

The set E is topologically closed: if a point x is not an eternal point, $x \notin E$, then there exists a neighborhood of x which also does not belong to E . This is so because the point x must have thermalized at a finite time t together with a horizon-size region around it.

The set E does not contain isolated points: in any neighborhood of a point $x \in E$ there exists, with probability 1, another point from E . To show this, suppose that a neighborhood of a point $x \in E$ contains no other points of E . Define t_0 to be the time when this neighborhood grows to horizon (or larger) size and let H_0 be the Hubble constant at point x at that time. Consider an infinite sequence of horizon-sized inflating regions R_n surrounding the point x at times $t_n = t_0 + nH_0^{-1}$, $n = 0, 1, 2, \dots$; by assumption, the evolution of the region R_n at each e -folding is such that R_n expands into e^3 inflating subdomains, of which one is to become R_{n+1} and all others contain no eternal points. There is a certain probability $p < 1$ for a horizon-sized region R_n containing one eternal point to have such an evolution during one e -folding; but the probability for *all* regions R_n , $n = 0, 1, 2, \dots$ to have the same evolution is the product of infinitely many factors p , which is zero.

By extension, any neighborhood of $x \in E$ contains, with probability 1, infinitely many other points from E .

Since E has measure zero, any given comoving point has *a priori* zero probability to belong to E . Therefore, any fixed line segment or surface has zero probability to consist entirely of points from E .

C. Gauge invariance of fractal dimension

Previous calculations [7, 10, 11] concerned the fractal dimension of the inflating domain at a finite time and assumed a certain time slicing (gauge). The results depended on the choice of the time coordinate (except for the case of Ref. [11] where expansion of inflating domains was homogeneous and all choices of time coordinate are equivalent). Here, instead of the set of points that are inflating at a given time, we consider the set E of eternal points as a subset of comoving spatial section at initial time. Now we shall show that the fractal dimension of the set E is invariant under smooth changes of spacetime coordinates.

The set E can be thought of as a locus of eternal points drawn in comoving spatial coordinates on the initial space-like slice (at $t = 0$). The construction of the set E requires to know whether a given point has thermalized at *any* finite time $t > 0$. Therefore the eternal set does not depend at all on the choice of time slicing of the spacetime to the future of the original slice. Its properties may, at most, be functions of the initial value of ϕ at $t = 0$.

Further, one can show that the fractal dimension of a set drawn in a finite region of space remains unchanged under any smooth change of coordinates in that region. First, the fractal dimension of a set is unchanged under a linear transformation of coordinates with a non-degenerate Jacobian J , because the coarse-grained volume $V(l)$ will simply gain a factor J and this will not change the fractal dimension defined by Eq. (6). Given an arbitrary smooth coordinate transformation, we can choose a sufficiently small scale l_0 at which the coordinates are locally changed by a constant linear transformation. We can divide the initial region into subdomains of size $l_1 < l_0$ and find $V(l)$ for $l < l_1$ by adding the volumes $V_i(l)$ calculated over these subdomains. Under the coordinate change, each volume $V_i(l)$ will be multiplied by a corresponding Jacobian J_i , but the Jacobian in a given subdomain does not depend on l for sufficiently small l_1 . Therefore, in the limit $l \rightarrow 0$ the dominant dependence of $V(l)$ on l of the form $V(l) \sim l^\gamma$ will be unchanged under coordinate transformations.

Finally, the fractal dimension of the eternal set E is the same when computed in an arbitrarily small neighborhood of any eternal point. This is because the set E has no isolated points and any point $x \in E$ is surrounded by infinitely many other eternal points lying arbitrarily close to x . The statement can be justified more formally as follows. In the next subsection we show (without using the present statement) that the fractal dimension of E is, in fact, independent of the initial value of ϕ in the initial horizon-sized region. If we choose a small comoving neighborhood $U(x)$ of an eternal point x , there will be a time t_1 when $U(x)$ grows to horizon size. Since $U(x)$ contains an eternal point x , it could not have thermalized by the time t_1 . Therefore at time t_1 the neighborhood $U(x)$ is undergoing inflation and it can be taken as the initial comoving region with a certain value of ϕ . The future evolution of any inflating horizon-sized region is statistically the same and therefore the fractal dimension of the subset of E computed in the neighborhood $U(x)$ starting at $t = t_1$ is the same as that of the full set E .

Unlike other fractals occurring in nature, the fractal set E has no short-distance cutoff: it retains its fractal structure at all scales (in comoving space).

D. Fractal dimension from Fokker-Planck equation

In this section we show that the fractal dimension of the eternal set E can be obtained from the comoving-volume probability distribution of the field ϕ in a special gauge. This probability distribution is a solution of the comoving-volume FP equation in that gauge. The dominant eigenvalue of this FP equation determines the fractal dimension of the set E and coincides with the dominant eigenvalue of the FP equation in the scale factor time variable.

We introduce a special time variable θ related to the scale factor a by

$$\theta(t, \mathbf{x}) = \ln a(t, \mathbf{x}) + \ln \frac{H(t, \mathbf{x})}{H_0}. \quad (7)$$

The meaning of this variable is the physical expansion scale $a(t, \mathbf{x})$ relative to the local horizon scale H^{-1} ; a surface of equal θ consists of spacetime points (t, \mathbf{x}) where the initial horizon size H_0^{-1} had expanded to e^θ times the current horizon size at the same point. One may call θ a “horizon scale factor” time variable.

The time variable θ is not strictly monotonic: $\dot{\theta} = \ddot{a}/\dot{a}$ (dots denote proper time derivatives) and if H is decreased rapidly enough so that the accelerated expansion locally stops, the “time” θ will decrease despite growth of the scale factor a . However, this is expected to happen only near the end of inflation, whereas in other regimes such large fluctuations of H are extremely rare and expansion is almost always accelerated. We shall assume that the time variable θ is well-behaved throughout the allowed interval of ϕ and truncate that interval near the end of inflation if necessary.

Below we shall see that in the gauge θ , the comoving volume probability distribution $P^{(\theta)}(\theta, \phi)$ at late times θ has the form

$$P^{(\theta)}(\theta, \phi) \approx \psi^{(\theta)}(\phi) e^{-\gamma\theta}, \quad (8)$$

where $(-\gamma)$ is the largest exponent dominating the solution at late times and $\psi^{(\theta)}$ is an appropriate eigenfunction. Assuming that Eq. (8) gives the asymptotic solution at late times, one finds the fractal dimension of the set E to be

$$\dim E = 3 - \gamma. \quad (9)$$

This result can be derived from Eq. (8) as follows. Let $d = 3$ be the dimension of space. An initial comoving region of horizon size with a constant field value ϕ_0 has linear

extent H_0^{-1} , where $H_0 = H(\phi_0)$, and we can choose comoving coordinates in the region to divide it into l^{-d} identical cubical subdomains of equal initial linear size lH_0^{-1} , with $l \ll 1$. At a sufficiently late time $\theta_1(l)$ when these comoving subdomains have grown to horizon size, there will be on average $l^{-d}P^{(\theta)}(\theta_1, \phi)d\phi$ subdomains still inflating with field value within the interval $[\phi, \phi + d\phi]$. (We assume that eternal inflation has been realized in the initial region, otherwise the number of inflating subdomains would be zero at sufficiently late times.) The probability for a horizon-size region with field value ϕ to contain at least one eternal point is a function of ϕ which we denote $X(\phi)$. It follows that at time θ_1 there will be, on average, $l^{-d} \int P^{(\theta)}(\theta_1, \phi) X(\phi) d\phi$ subdomains that contain an eternal point. Since $e^{\theta_1} = l^{-1}$ and θ_1 is large for small l , we can use Eq. (8) to obtain the total (comoving) volume of those subdomains,

$$V(l) = e^{-\gamma\theta_1} \int \psi^{(\theta)}(\phi) X(\phi) d\phi = \text{const} \cdot l^\gamma. \quad (10)$$

This is the coarse-grained volume of the eternal set at scale l . By definition of fractal dimension, $\dim E = d - \gamma$.

The constant in Eq. (10) is nonzero if $X(\phi) \neq 0$ and if the set E is not empty; otherwise we cannot assume that there are many inflating subdomains and estimate $V(l)$ by its average value, as we have done in Eq. (10). This constant also absorbs all dependence on the initial value of ϕ in the initial comoving region. It follows that the fractal dimension is independent of the initial value of ϕ and is well-defined as long as eternal inflation is realized in the domain of consideration.

Now we need to determine γ . Instead of using a Fokker-Planck equation in the gauge θ , we will show that γ is equal to the dominant eigenvalue of the Fokker-Planck equation in the scale factor time variable. This will give a computational prescription to obtain the fractal dimension of the set E in a given model of inflation.

The distribution of volume $P^{(\theta)}(\theta, \phi)$ in the time gauge θ is related to the distribution $P(t^{(0)}, \phi)$ in the scale factor time gauge $t^{(0)} \equiv \ln a$ by

$$P(t^{(0)}, \phi) d\phi = P^{(\theta)}\left(t^{(0)} + \ln \frac{H(\phi)}{H_0}, \phi\right) d\phi. \quad (11)$$

To justify this, consider the combined comoving volume of all domains in which the value of the field ϕ is between ϕ_1 and $\phi_1 + d\phi$ and scale factor is $\exp[t^{(0)}]$. This volume is $P(t^{(0)}, \phi_1)d\phi$. [The distribution $P(t^{(0)}, \phi)$ as obtained from the FP equation is not normalized since $\int P(t^{(0)}, \phi)d\phi < 1$ is the total comoving volume of inflating regions in units

of the initial comoving volume.] Throughout all volume characterized by ϕ_1 and $t^{(0)}$, the “horizon scale factor” θ has the value $\theta(t^{(0)}, \phi_1) = t^{(0)} + \ln[H(\phi_1)/H_0] + O(d\phi)$. Also, any volume in which $\theta = \theta(t^{(0)}, \phi_1)$ and $\phi \in [\phi_1, \phi_1 + d\phi]$ must have scale factor $\exp[t^{(0)}] + O(d\phi)$. Therefore, $P(t^{(0)}, \phi_1)d\phi$ is, up to $O(d\phi^2)$, equal to the total volume of all domains in which $\theta = \theta(t^{(0)}, \phi_1)$ and $\phi \in [\phi_1, \phi_1 + d\phi]$. This justifies Eq. (11).

The same justification can be given for an analogous relation between physical-volume distributions in gauges θ and $t^{(0)}$, since the argument does not rely on the comoving nature of the volume to which the distributions relate.

The FP equation for the comoving volume distribution in the scale factor time gauge $t^{(0)}$ is

$$\frac{\partial P}{\partial t^{(0)}} = \frac{\partial^2}{\partial \phi^2} (D^{(0)} P) - \frac{\partial}{\partial \phi} (v^{(0)} P). \quad (12)$$

Here

$$D^{(0)} = \frac{H^2}{8\pi^2}, \quad v^{(0)} = -\frac{1}{4\pi H} \frac{dH}{d\phi} \quad (13)$$

are the kinetic coefficients in this gauge. At late times $t^{(0)}$, the solution of Eq. (12) is

$$P(t^{(0)}, \phi) \approx e^{-\gamma t^{(0)}} \psi(\phi), \quad t^{(0)} \gg 1, \quad (14)$$

where γ is the dominant eigenvalue of Eq. (12). Late times $t^{(0)}$ correspond also to large values of θ since H/H_0 is bounded. It follows from Eq. (11) that the distribution $P^{(\theta)}$ has the asymptotic form

$$P^{(\theta)}(\theta, \phi) \approx e^{-\gamma \theta} \left(\frac{H(\phi)}{H_0} \right)^{-\gamma} \psi(\phi), \quad \theta \gg 1. \quad (15)$$

Therefore, the dominant eigenvalue γ of the FP equation in the scale factor time gauge is the relevant quantity for the fractal dimension in Eq. (9).

This result shows that the fractal dimension of the set E given by Eq. (9) is the same as the fractal dimension d_{fr} of inflating domain [7, 10, 11] if the latter is computed on surfaces of equal scale factor. Although γ is an eigenvalue of the FP equation in a particular gauge, the quantity $d - \gamma$ has a physical interpretation as the fractal dimension of a set defined independently of spacetime coordinates; eigenvalues of the FP equation in other gauges do not seem to have a gauge-invariant interpretation.

III. CONDITIONS FOR ETERNAL INFLATION

We have distinguished two issues regarding the possibility of eternal inflation: first, eternal inflation may be entirely disallowed in certain models; and second, in models where eternal inflation is possible, it may accidentally not have realized in certain domains. In this section, we first examine conditions on an inflationary model which make eternal inflation possible, and then we shall obtain the probability for eternal inflation to be realized in a given comoving region. The condition that the latter probability does not vanish is another criterion for presence of eternal inflation. Both criteria are shown to be independent of the choice of spacetime coordinates. Finally, we demonstrate that the two criteria for presence of eternal inflation are consistent with each other.

A. Presence of eternal inflation

The basic condition for possibility of eternal inflation in an inflationary model is $\gamma_V > 0$, where γ_V is the dominant eigenvalue of Eq. (1). However, this equation itself and the value of γ_V are gauge-dependent. We shall now show that $\gamma_V > 0$ is in fact a gauge-invariant condition.

The key idea of the proof is that if $\gamma_V = 0$ in some gauge it must also be equal to zero in all other gauges, and therefore γ_V cannot be positive in one gauge and negative in another.

Suppose that there exist two time variables, t_0 and t_1 such that the dominant eigenvalue γ_V has opposite sign in the gauges defined by t_0 and t_1 . Introduce a one-parametric family of time variables t_α , $0 \leq \alpha \leq 1$ to interpolate between these time variables,

$$\frac{dt_\alpha}{dt_0} \equiv \left(\frac{dt_1}{dt_0} \right)^\alpha. \quad (16)$$

The new time variables t_α are defined by integrating Eq. (16) along comoving world-lines. The dominant eigenvalue γ_V of the FP equation in gauge t_α is a function of α . This function $\gamma_V(\alpha)$ is continuous, because the differential operator in the FP equation is self-adjoint, its dominant eigenvalue γ_V is non-degenerate [9] and under a small change of α this eigenvalue must change by a small amount, found from standard perturbation theory. By continuity it follows that there exists a value α_0 such that $\gamma_V(\alpha_0) = 0$. The FP equation in this time

parametrization has a stationary solution $P_V^{(0)}$,

$$\frac{\partial^2}{\partial \phi^2} (D_\alpha P_V^{(0)}) - \frac{\partial}{\partial \phi} (v_\alpha P_V^{(0)}) + 3H_\alpha P_V^{(0)} = 0. \quad (17)$$

Here D_α , v_α and H_α are kinetic coefficients in the gauge t_α . They differ from $D_0(\phi)$, $v_0(\phi)$ and $H_0(\phi)$ in the gauge t_0 by the factor $(dt_\alpha/dt_0)^{-1}$. The factor dt_α/dt_0 can be absorbed into $P_V^{(0)}$, and we obtain a stationary solution of the FP equation in all other gauges t_α . Existence of this solution with eigenvalue 0 in all gauges t_α contradicts our assumption that the largest eigenvalue of the FP equation is negative in one of the gauges t_0 or t_1 .

We have found that the dominant eigenvalue γ_V of Eq. (1) must have the same sign in all gauges and, therefore, presence or absence of eternal inflation can be judged by the sign of γ_V in any gauge.

B. A diffusion equation for $X(\phi)$

Assuming now that $\gamma_V > 0$, we consider the probability for eternal inflation to be realized in a given region. In Sec. II D, we have denoted by $X(\phi)$ the probability for an initial inflating region of one horizon size, in which the scalar field has value ϕ , to contain at least one eternal point. It will be convenient to work with the complementary probability $\bar{X}(\phi) = 1 - X(\phi)$. In this section we find that $\bar{X}(\phi)$ satisfies a nonlinear branching diffusion equation [Eq. (23)] that resembles the backward FP equation for the physical volume probability distribution. We derive this equation and show its independence of time parametrization.

Here we shall use proper time t as the time variable. The random walk of the inflaton field ϕ is described by a Langevin equation which can be written as a difference equation [5],

$$\phi(t + \Delta t) = \phi(t) + v(\phi) \Delta t + \xi \sqrt{2D(\phi) \Delta t}, \quad (18)$$

where $\xi = \xi(t, \mathbf{x})$ is a normalized random “noise” representing fluctuations, and $v(\phi)$, $D(\phi)$ are the kinetic coefficients given by Eq. (2). The evolution of ϕ in a horizon-size domain with field value $\phi = \phi_0$ at time $t = 0$ gives rise to independent random walks of ϕ in each of the “daughter” horizon-size subdomains that were formed out of the original domain. After one e -folding, *i.e.* after time $\Delta t = H^{-1}$, there are $N \equiv \exp(3H\Delta t) \approx 20$ daughter subdomains of approximately horizon size. The stochastic process corresponding to ξ assigns a probability density $P(\xi_1, \dots, \xi_N) d\xi_1 \dots d\xi_N$ for various sets of values of ξ in the N daughter subdomains.

The probability $\bar{X}(\phi_0)$ to have no eternal points in the region is equal to the probability to have no eternal points in any of its N daughter subdomains. The evolution in inflating daughter subdomains proceeds independently, so the probability to have no eternal points in each of them is described by the same function $\bar{X}(\phi)$ evaluated at $\phi(t + \Delta t)$. (Although the daughter subdomains have slightly varying $H(\phi)$ and are not exactly of horizon size, the correction due to this is negligible.) This gives an integral equation,

$$\begin{aligned} \bar{X}(\phi_0) &= \int d\xi_1 \dots d\xi_N P(\xi_1, \dots, \xi_N) \\ &\times \prod_{i=1}^N \bar{X}\left(\phi_0 + v(\phi_0)\Delta t + \xi_i \sqrt{2D(\phi_0)\Delta t}\right). \end{aligned} \quad (19)$$

Here we use the Ito interpretation of the Langevin equation, where v and D are evaluated at the initial point of the step, $\phi = \phi_0$.

Since correlations between different daughter subdomains are small [21], we can approximate the probability density of ξ_i by a product of independent identical distributions, $P(\xi_1, \dots, \xi_N) = P(\xi_1) \dots P(\xi_N)$. Then Eq. (19) becomes

$$\begin{aligned} \bar{X}(\phi_0) &= \\ &\left[\int d\xi P(\xi) \bar{X}\left(\phi_0 + v\Delta t + \xi\sqrt{2D\Delta t}\right) \right]^{\exp(3H\Delta t)}. \end{aligned} \quad (20)$$

Rather than trying to solve Eq. (20), we consider its limit for small values of $H\Delta t$. Although the Langevin description of the random walk given by Eq. (18) is valid only for times $H\Delta t \gtrsim 1$, we shall formally consider it to be valid at all values of Δt and take a partial derivative $\partial/\partial\Delta t$ of Eq. (20) at $\Delta t = 0$. The same formal limit $\Delta t \rightarrow 0$ is used to derive the FP equations such as Eqs. (1), (12) in the standard stochastic formalism of inflation, and the same limits of validity apply to the new diffusion equation [Eq. (23)] that will result from the present argument.

We find

$$\begin{aligned} \frac{\partial \bar{X}(\phi_0)}{\partial \Delta t} &= 0 = 3H\bar{X} \ln \bar{X} \\ &+ \lim_{\Delta t \rightarrow 0} \int d\xi P(\xi) \frac{d\bar{X}(\phi_0 + v\Delta t + \xi\sqrt{2D\Delta t})}{d\phi} \\ &\times \left(v + \frac{\xi\sqrt{2D}}{2\sqrt{\Delta t}} \right). \end{aligned} \quad (21)$$

We now expand $d\bar{X}/d\phi$ in Taylor series around $\phi = \phi_0$ and use the fact that ξ is a normalized random variable,

$$\int d\xi P(\xi) = 1, \quad \langle \xi \rangle = 0, \quad \langle \xi^2 \rangle = 1, \quad (22)$$

to obtain an equation for \bar{X} ,

$$D \frac{d^2 \bar{X}}{d\phi^2} + v \frac{d\bar{X}}{d\phi} + 3H\bar{X} \ln \bar{X} = 0. \quad (23)$$

This equation needs to be supplemented with suitable boundary conditions at end of inflation and/or at Planck boundaries. The probability \bar{X} to have no eternal points for a region which is near the end of inflation $\phi = \phi_*$ should be 1. Domains reaching Planck boundaries ϕ_{Pl} will never thermalize and we could set $\bar{X} = 0$ at those boundaries; this is consistent with the viewpoint that super-Planck domains effectively disappear [7]. Physically meaningful solutions of Eq. (23) should vary between 0 and 1. Therefore, the additional requirements are

$$0 \leq \bar{X} \leq 1; \quad \bar{X}(\phi_*) = 1; \quad \bar{X}(\phi_{Pl}) = 0. \quad (24)$$

It is straightforward to verify that Eq. (23), unlike Eq. (1), is a gauge-invariant equation. A change of time variable, $t \rightarrow t'$, with $dt'/dt = T(t, \phi)$, will divide the functions $D(\phi)$ and $v(\phi)$, as well as the factor $3H(\phi)$ in the “growth” term, by $T(t, \phi)$. This extra factor $T(t, \phi)$ cancels and Eq. (23) remains unchanged. This is to be expected since the probability $\bar{X}(\phi)$ is defined in a gauge-invariant way and should be described by a gauge-invariant equation.

Equation (23) is a branching diffusion equation similar to the backward FP equation for the physical volume probability distribution in the Ito factor ordering, except for the extra factor $\ln \bar{X}$ that makes it nonlinear. An analogous diffusion equation can be derived for models of inflation with multiple scalar fields ϕ_k ,

$$D \frac{\partial}{\partial \phi^k} \frac{\partial \bar{X}}{\partial \phi_k} + v_k \frac{\partial \bar{X}}{\partial \phi_k} + 3H\bar{X} \ln \bar{X} = 0, \quad (25)$$

where

$$v_k(\phi) \equiv -\frac{1}{4\pi} \frac{\partial H}{\partial \phi^k} \quad (26)$$

are the new drift coefficients (in the proper time gauge). Boundary conditions will be $\bar{X}(\phi) = 1$ along thermalization boundaries and $\bar{X}(\phi) = 0$ along Planck boundaries (if any).

The qualitative behavior of the probability $\bar{X}(\phi)$ for a chaotic type inflationary model is illustrated in Fig. 1. An approximate WKB solution in the fluctuation-dominated regime

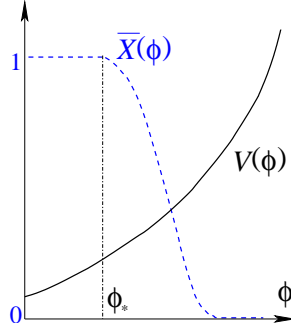


FIG. 1: Probability $\bar{X}(\phi)$ to have no eternal points.

can be obtained if we disregard the drift term $v(\phi) \bar{X}'$ and write an ansatz

$$\bar{X}(\phi) = Ae^{-W(\phi)}, \quad (27)$$

where A is approximately constant and $W(\phi)$ is a slow-changing function, $|W''| \ll (W')^2$ that satisfies

$$D(W')^2 - 3HW = 0. \quad (28)$$

The solution is

$$W(\phi) = 6\pi \left[\int_{\phi_q}^{\phi} \frac{d\phi}{H(\phi)} \right]^2 \quad (29)$$

where ϕ_q is the boundary of fluctuation-dominated range of ϕ . It shows that the probability to have eternal points $1 - \bar{X}(\phi)$ exponentially rapidly approaches 1 in the fluctuation-dominated regime.

Since $\bar{X} > 0$, there is always a certain (perhaps exceedingly small) probability for eternal inflation to not be realized within a given initial horizon-sized region. On the other hand, an inflationary model could be fine-tuned to entirely disallow eternal inflation, so that Eqs. (23)-(24) have no solutions other than $\bar{X} \equiv 1$. Note that, in the absence of Planck boundaries, $\bar{X} \equiv 1$ is always a solution of Eqs. (23)-(24). This solution would mean that there are no eternal points anywhere. Equations (23)-(24) alone do not provide enough information to choose between the constant solution $\bar{X} \equiv 1$ and a nontrivial solution $\bar{X}(\phi)$. However, we know from considerations in Sec. II that, if eternal inflation is allowed in a model, then eternal points are possible, *i.e.* the probability to have eternal points is nonzero. Therefore, we should choose the solution $\bar{X} \equiv 1$ only when no other solution of Eq. (23) that satisfies Eq. (24) can be found.

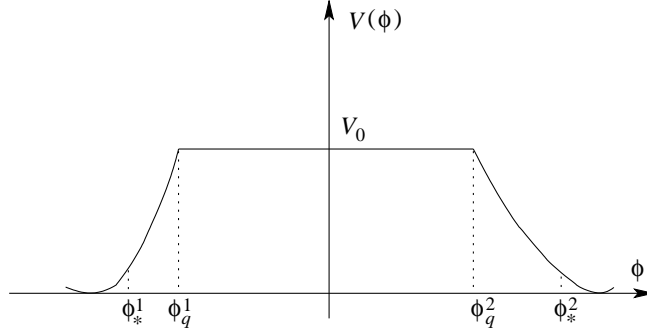


FIG. 2: Illustrative potential for eternal inflation.

C. An exact solution

For illustrative purposes, we solve Eq. (23) explicitly for a flat potential connected to two slow-roll slopes with negligible diffusion (Fig. 2). This potential has been considered in Ref. [19] as a qualitative illustration of eternal inflation. The allowed range of ϕ is divided into two regimes: in the fluctuation-dominated regime, $\phi_q^1 < \phi < \phi_q^2$, the potential is flat and we take $H(\phi) = H_0$, $D(\phi) = D_0$ and $v(\phi) = 0$. In the second regime ($\phi_*^1 < \phi < \phi_q^1$ or $\phi_q^2 < \phi < \phi_*^2$), we take $D(\phi) = 0$ and $v(\phi) \neq 0$, to represent pure deterministic motion without fluctuations.

We expect that $\bar{X} \equiv 1$ in the second regime, since all regions thermalize within finite time. In this regime Eq. (23) becomes

$$v\bar{X}' + 3H\bar{X} \ln \bar{X} = 0. \quad (30)$$

The general solution of this equation is

$$\bar{X}(\phi) = \exp \left[-\frac{C}{a(\phi)^3} \right], \quad (31)$$

where

$$a(\phi) \equiv \exp \left[-4\pi \int_{\phi_*}^{\phi} \frac{H d\phi}{H'} \right] \quad (32)$$

is the scale factor along a slow roll trajectory, and C is an integration constant. The boundary condition $\bar{X}(\phi_*) = 1$ forces $C = 0$ and therefore $\bar{X} \equiv 1$ throughout the deterministic region, as expected.

Turning now to the fluctuation-dominated regime, we need to solve

$$D_0 \bar{X}'' + 3H_0 \bar{X} \ln \bar{X} = 0 \quad (33)$$

with boundary conditions

$$\bar{X}(\phi_q^{1,2}) = 1. \quad (34)$$

For simplicity, we take $\phi_q^2 \equiv \phi_q = -\phi_q^1$. Equation (33) describes a Hamiltonian system of a particle with mass D_0 moving in “time” ϕ in a potential

$$U(\bar{X}) = \frac{3H_0}{4}\bar{X}^2(2\ln\bar{X} - 1). \quad (35)$$

This potential monotonically decreases from $U = 0$ at $\bar{X} = 0$ to its minimum $U = -3H_0/4$ at $\bar{X} = 1$ (see Fig. 3). Boundary conditions correspond to a “trajectory” $\bar{X}(\phi)$ starting and ending at $\bar{X} = 1$ such that the total “travel time” is $2\phi_q$. Such a trajectory is unique and is characterized by the lowest reached value \bar{X}_0 (the turning point) such that

$$\int_{\bar{X}_0}^1 \frac{dx\sqrt{D_0}}{\sqrt{2U(\bar{X}_0) - 2U(\bar{X})}} = \phi_q. \quad (36)$$

The value \bar{X}_0 is the lowest probability to have no eternal points and is naturally achieved at $\phi = 0$, in the middle of the fluctuation-dominated range of ϕ . In this model, \bar{X}_0 is a function of (ϕ_q/H_0) which can be obtained numerically. For $\phi_q \gtrsim H_0$, one can approximate $U(\bar{X}_0) \approx 0$ and obtain

$$\bar{X}_0 \sim 2 \exp\left[-\frac{1}{2}s^2 - s\right], \quad s \equiv 2\pi\sqrt{3}\frac{\phi_q}{H_0} \gtrsim 1. \quad (37)$$

Numerical evaluation shows that this asymptotic expression holds to 5% accuracy for $2\pi\phi_q/H_0 > 2$ which corresponds to $\bar{X}_0 < 10^{-4}$. As expected, the probability $X = 1 - \bar{X}$ to have eternal points is very close to 1 for regions that start in the fluctuation-dominated regime.

The opposite case, $\bar{X}_0 \approx 1$, corresponds to motion near the minimum of the potential $U(\bar{X})$. Since the period of such motion is approximately independent of amplitude, Eq. (36) cannot be satisfied unless

$$\phi_q > \phi_c \equiv \frac{H_0}{\sqrt{96}}. \quad (38)$$

For $\phi_q \leq \phi_c$, Eq. (33) has no solutions except $\bar{X} \equiv 1$. This indicates absence of eternal inflation when the fluctuation-dominated range of ϕ is sufficiently narrow.

It is straightforward to check that for the inflaton potential $V(\phi)$ of Fig. 2, Eq. (1) with boundary conditions $P_V(\phi_{\pm q}, t) = 0$ admits a non-negative eigenvalue γ_V if and only if Eq. (38) holds. In the next subsection we shall see that this result is not a coincidence.

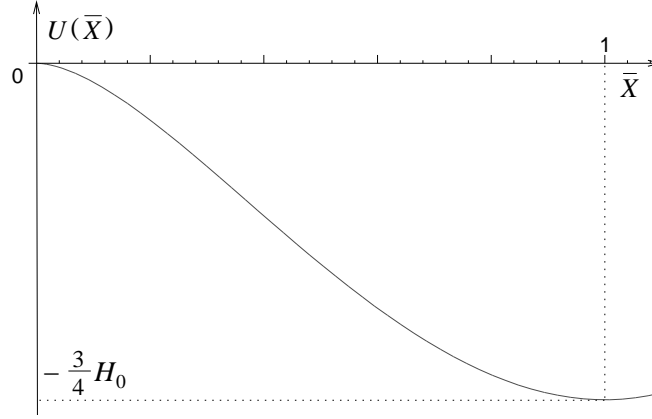


FIG. 3: Effective potential of Eq. (35).

D. Consistency of conditions for eternal inflation

We have given two gauge-invariant criteria for presence of eternal inflation: positivity of the dominant eigenvalue γ_V of Eq. (1) and existence of a non-trivial solution of Eqs. (23)-(24). Since the relevant diffusion equations are somewhat dissimilar, a natural question is whether these two criteria are equivalent. A positive answer is suggested by the analytic example of the previous section. Here we shall see that Eqs. (23)-(24) cannot have a nontrivial solution $\bar{X}(\phi) \not\equiv 1$ if the FP equation has a non-positive dominant eigenvalue.³

The idea of the proof is to consider a suitable non-negative function of ϕ , integrate it over ϕ and obtain $\gamma_V > 0$ as a result.

Let \mathcal{L} be the differential operator of Eq. (1),

$$\mathcal{L}P_V \equiv \frac{\partial^2}{\partial \phi^2} (DP_V) - \frac{\partial}{\partial \phi} (vP_V) + 3HP_V. \quad (39)$$

The spectrum of eigenvalues of \mathcal{L} is bounded from above, and the eigenfunction $\psi_V(\phi)$ of Eq. (3) corresponding to the largest eigenvalue γ_V is everywhere positive, as a ground state of a self-adjoint operator [9]. Suppose that a non-trivial solution $\bar{X}(\phi) \not\equiv 1$ of Eqs. (23)-(24) exists; then the following integral must be positive,

$$I_1 \equiv \int_{\phi_1}^{\phi_2} (1 - \bar{X}(\phi)) \psi_V(\phi) d\phi > 0. \quad (40)$$

³ I do not yet have a proof of existence of a nontrivial solution $\bar{X}(\phi)$ when $\gamma_V > 0$, which is needed to rigorously demonstrate equivalence of these two criteria.

Here $\phi_{1,2}$ are left and right thermalization boundaries, $\phi_2 > \phi_1$. Since

$$\gamma_V = \frac{1}{I_1} \int_{\phi_1}^{\phi_2} (1 - \bar{X}) \mathcal{L}\psi_V d\phi, \quad (41)$$

we would obtain the desired inequality $\gamma_V > 0$ if we prove that

$$I_2 \equiv \int_{\phi_1}^{\phi_2} (1 - \bar{X}) \mathcal{L}\psi_V d\phi > 0. \quad (42)$$

Integrating Eq. (42) by parts and using boundary conditions $\bar{X}(\phi_{1,2}) = 1$, we find

$$\begin{aligned} I_2 &= \int_{\phi_1}^{\phi_2} \psi_V \cdot \mathcal{L}^* [1 - \bar{X}] d\phi \\ &\quad + D(\phi) \psi_V \frac{d\bar{X}}{d\phi} \Big|_{\phi_1}^{\phi_2}. \end{aligned} \quad (43)$$

Here \mathcal{L}^* is the operator conjugate to \mathcal{L} . Using Eq. (23), we obtain

$$\begin{aligned} &\mathcal{L}^* [1 - \bar{X}] \\ &= -D \frac{\partial^2 \bar{X}}{\partial \phi^2} - v \frac{\partial \bar{X}}{\partial \phi} + 3H (1 - \bar{X}) \\ &= 3H (1 - \bar{X} + \bar{X} \ln \bar{X}) \geq 0. \end{aligned} \quad (44)$$

The integral $\int \psi_V \mathcal{L}^* [1 - \bar{X}] d\phi$ is positive because its integrand is nonnegative and not everywhere zero. The boundary term $D\psi_V \bar{X}'$ is non-negative at ϕ_2 and non-positive at ϕ_1 . This proves Eq. (42) and therefore γ_V must be positive.

It follows that if the dominant eigenvalue γ_V is zero or negative, a nontrivial solution $\bar{X}(\phi) \neq 1$ cannot exist.

IV. MERGING OF THERMALIZED REGIONS

In the previous section, we found a function $X(\phi)$ that describes the probability to realize eternal inflation. Here we show that the same function gives bounds on probabilities of other global properties of the set E . In particular, we consider the issue of whether all thermalized domains ultimately merge and become connected to each other.

A. Global properties of the eternal set

Observe that the nonlinear diffusion equation for $\bar{X}(\phi)$ was derived using the argument that an inflating domain contains no eternal points if and only if none of its daughter

domains, after one e -folding, contain eternal points. It is clear that the same derivation would apply for the probability $Y(\phi)$ to realize some other property Y of the set E , as long as that property holds for a given horizon-size inflating domain if and only if it holds for all daughter domains, and given that the property Y holds for a thermalized domain ($Y(\phi_*) = 1$). These conditions define a certain class G of “global” properties that are binary alternatives, functions of the subset of E inside a given domain.

We find that the probability to realize any property from this class G is described by Eqs. (23)-(24). The property for the set E to be empty, which occurs with probability $\bar{X}(\phi)$, belongs to this class; other examples would be the property for the set E to not contain any continuous line segments, or any continuous 2-dimensional surface segments. Let us denote by \bar{S}_n the property of the eternal set to contain no fragments of continuous n -dimensional submanifolds, $n = 0, 1, 2, 3$.

The universal applicability of Eq. (23) to all global properties of class G can be interpreted as a statement that probabilities for global properties of the fractal set E must be “fixed points” under the “renormalization” (rescaling) of the set, and all such fixed points must satisfy Eq. (23).

In presence of eternal inflation, as we have seen, Eqs. (23)-(24) admit two solutions, namely $\bar{X} \equiv 1$ and $\bar{X} = \bar{X}(\phi) \neq 1$. Equations (23)-(24) alone do not give enough information for us to select one of these two solutions. To find the probability distribution $Y(\phi)$ for a particular property Y , other considerations specific to that property and to the given inflationary model are needed. However, we can conclude *a priori* that one of these two alternatives must be realized for any given property Y from the class G . This implies that a property from the class G either always holds, $Y(\phi) \equiv 1$, or holds with probability $Y(\phi) = \bar{X}(\phi)$. (Here $\bar{X}(\phi)$ is a fixed function, independent of the property Y .) For instance, it would be enough to show that $Y(\phi) > \bar{X}(\phi)$ for some property Y to conclude that $Y(\phi) \equiv 1$.

Since the same function $\bar{X}(\phi)$ describes the probability to have no eternal points, a natural question to ask is whether the occurrence of a given property in a comoving domain is correlated with presence of eternal points in the same domain. Consider a property Y of class G and suppose that we know that Y does not always hold; it follows that the property Y holds with the probability $Y(\phi) = \bar{X}(\phi) \neq 1$ in a horizon-size domain with value ϕ of the inflaton field. If a horizon-size domain has no eternal points (property \bar{X}), then the

property Y must hold in this domain. Therefore the probability for Y to hold if there are no eternal points is $Prob(\bar{X} \cap Y) = Prob(\bar{X}) = Prob(Y)$, while the probability for Y to hold in presence of eternal points is $Prob(X \cap Y) = Prob(Y) - Prob(\bar{X} \cap Y) = 0$.

We find that, for any property Y of class G , one of the following two alternatives must hold (in a given inflationary model): either Y is always true for any horizon-size domain, or Y is true if and only if the domain contains no eternal points. In other words, a property Y of class G either always holds or is perfectly anti-correlated with presence of eternal points and, by extension, perfectly correlated with other properties of class G . For example, the property \bar{S}_0 holds in a domain only when the domain contains no eternal points, while \bar{S}_3 always holds. In principle, details of the model should determine which of these alternatives is realized for a particular property Y .

B. Bounds on merging of thermalized regions

In this section, we give a bound on merging of thermalized regions based on the fractal dimension of the eternal set E .

The set of all comoving points that will eventually thermalize is a complement of the set E . Thermalized regions will not merge if the set E encloses pockets of thermalized comoving space. To enclose a region of a 3-dimensional space, the set E must contain a continuous boundary of the enclosure which is at least a 2-dimensional surface, so the fractal dimension of E must be at least 2. If the fractal dimension $\dim E$ of the set E turns out to be less than 2, all thermalized domains must be merged. If, on the other hand, $\dim E > 2$, then the question remains unresolved, because there exist sets of fractal dimension up to 3 which nevertheless do not enclose any interior domains.

The condition $\dim E < 2$ gives a topological bound on the possibility of merging of thermalized regions. However, this bound may be rather weak because in typical inflationary scenarios $\dim E = 3 - \gamma$ with $\gamma \ll 1$. For instance, in a potential with a flat maximum at $\phi = 0$, an estimate of Ref. [9] gives

$$\gamma \approx \frac{1}{16\pi} \frac{|V''(0)|}{V(0)} \ll 1. \quad (45)$$

In this case the condition $\gamma > 1$ is not satisfied and the question of merging remains open.

One may distinguish *merging* of all thermalized regions into one connected domain from

percolation of thermalized regions, that is, formation of at least one infinitely large thermalized cluster (of infinite comoving volume). Percolation is a weaker condition, since merging entails percolation but not vice versa. Guth and Weinberg [12] have considered the “old” scenario of inflation with bubble nucleation and gave bounds on the nucleation rate ε per Hubble 4-volume for the bubbles (representing thermalized regions) to percolate. They have found that $\varepsilon \leq 1.1 \times 10^{-6}$ is a sufficient condition for non-percolation of bubbles, while $\varepsilon \geq 0.24$ guarantees percolation. In the model of bubble nucleation, the fractal dimension of the inflating domain at constant proper time is [11]

$$d_{fr} = 3 - 4\pi\varepsilon/3. \quad (46)$$

Since in this model the expansion of inflating domains is homogeneous with $H = \text{const}$, time parametrization by scale factor is equivalent to proper time parametrization, and the fractal dimension of the set E has the same value, $\dim E = d_{fr}$. It is interesting to note that the sufficient condition given in Ref. [12] for percolation of bubbles, $\varepsilon \geq 0.24$, approximately coincides (to 2 decimal digits) with the topological bound $d_{fr} < 2$ which guarantees merging.⁴ I conjecture that, in models of scalar field inflation, an upper bound on the merging probability may be established using the fractal dimension of the set E : namely, merging does not occur if $\dim E$ is sufficiently close to 3.

The property that all regions that eventually thermalized (all points outside the set E) are merged into one connected region is not a property of class G , because merging of thermalized regions in a domain does not entail merging in all of its subdomains (it is possible that some subdomains are divided by fragments of eternally inflating walls which, nevertheless, do not globally enclose an interior region). We can consider a weaker property that belongs to the class G : the property \bar{S}_2 for a region to contain no eternally inflating 2-dimensional surfaces. This property is sufficient (but not necessary) for merging. On the other hand, if a domain has a nonzero probability to contain fragments of 2-dimensional eternally inflating surfaces, it will contain, with the same probability, an infinite number of arbitrarily small such fragments. If this probability has been realized, it is likely (but not certain) that merging of thermalized regions will be impossible in that domain.

From considerations in the previous section we may conclude that, in a given inflationary model, either the property \bar{S}_2 always holds (and as a consequence all thermalized domains

⁴ I am grateful to A. Vilenkin for bringing this point to my attention.

will always merge), or the property \bar{S}_2 never holds in regions where eternal points are present. This would provide another bound for the probability of merging if the probability to realize the property \bar{S}_2 were determined.

I have been able to show that in any model of scalar field inflation described by Eq. (1), the eternal set E cannot contain any differentiable line segments or fragments of differentiable surfaces (details of the proof will be given elsewhere). The property of E to not contain any differentiable line segments also belongs to the class G and its probability can be shown to exceed $\bar{X}(\phi)$; then it follows from considerations of the previous section that this property must hold for all regions. However, merging of thermalized domains does not follow from this result, because the fractal set E may well contain continuous but not differentiable lines or surfaces.

V. SUMMARY AND DISCUSSION

In this work I have explored the fractal geometry of an eternally inflating universe at very large scales, in the formalism of stochastic inflation. I defined a fractal set E consisting of all eternally inflating points in comoving space. The definition of set E is independent of the choice of spacetime coordinates. Using the Fokker-Planck equation for comoving volume, I found the fractal dimension of the set E . It coincides with the fractal dimension d_{fr} of the inflating domain as defined in Ref. [10], if the latter were computed on surfaces of equal scale factor instead of equal proper time. In this way, the gauge-invariant construction of the fractal set E resolves the issue of gauge dependence that plagued previous calculations of the fractal dimension.

In a recent work, Bousso [22] considered infinite self-reproduction (proliferation) of de Sitter space due to nucleation of black holes. This effect may also create a fractal structure of the spacetime on very large scales. Although the proliferation effect involves topology change which is absent in the usual picture of eternal inflation, the fractal structure of the resulting spacetime can be investigated using the methodology developed above.

I have also examined the conditions for presence of eternal inflation in a given inflationary model. The original criterion was an unbounded growth of physical volume of inflating domains, which is equivalent to the condition $\gamma_V > 0$ imposed on the largest eigenvalue γ_V of the FP equation for the physical volume [Eq. (1)]. I have demonstrated that this condition

is gauge-invariant by proving that the sign of the dominant eigenvalue γ_V of Eq. (1) must be the same in all gauges. Then I found the probability $X(\phi)$ for an initial horizon-size region with scalar field value ϕ to contain at least one eternal point. This probability is a solution of a (gauge-invariant) nonlinear diffusion equation [Eq. (23)] derived in Sec. IIIB and analyzed in the following sections. To show that the new nonlinear diffusion equation is consistent with the existing formalism, I checked that the probability $X(\phi)$ vanishes when eternal inflation is not present, in agreement with the criterion $\gamma_V > 0$.

Finally, I have investigated the issue of merging of thermalized regions into one connected domain, as observed at arbitrarily late times. I obtained a topological bound on merging: all thermalized regions are guaranteed to merge if the fractal dimension of the set E is less than 2. This bound agrees with the previously found sufficient condition $\varepsilon \geq 0.24$ [12] for percolation in the model of bubble nucleation (in that model, Eq. (46) relates the nucleation rate ε to the fractal dimension of the set E). In Ref. [12], a sufficient condition for non-percolation, $\varepsilon \leq 1.1 \times 10^{-6}$, was also found. This suggests a possibility of a stronger bound on merging, namely that merging does not occur if the fractal dimension is sufficiently close to 3.

The stochastic formalism also constrains the probability of merging in a different way. Since the fractal structure of the eternally inflating set is preserved on all scales, the probability to realize any global property of the set must be invariant under rescaling. Equation (23) expresses the scale invariance of the probability $\bar{X}(\phi)$ for a horizon-size region to thermalize, and the same equation applies to probabilities to realize other global properties. I have defined a suitable class G of global properties described by Eq. (23); for example, the property \bar{S}_2 for a domain to contain no eternally inflating fragments of 2-dimensional surfaces is a property of class G . The property \bar{S}_2 is a sufficient condition for merging of thermalized domains (which is itself not a property of class G). I found that the probability to realize a property of class G is either identically equal to 1 or is described by the function $\bar{X}(\phi)$. This provides a constraint on the probability of merging. However, Eq. (23) alone does not provide enough information to resolve this alternative. This is to be expected, because the FP equation concerns only the total volume of space with given values of ϕ and is ignorant of the spatial distribution and shapes of individual thermalized or inflating domains. One needs to use other considerations, specific to a particular inflationary model, to select the correct solution.

Numerical simulations of chaotic inflationary models [7] suggested that thermalized regions always merge. However, because of the exponentially growing scales, it is difficult to accurately simulate the global structure of an eternally inflating spacetime, especially with realistic model parameters. An unambiguous resolution of the question of merging in a general scalar-field inflationary model requires further study.

Acknowledgments

The author is grateful to Alex Vilenkin for continued encouragement, fruitful conversations and valuable comments on the manuscript, and to Xavier Siemens and Vitaly Vanchurin for helpful discussions. This work was supported by the National Science Foundation.

-
- [1] A. Vilenkin, Phys. Rev. D **27**, 2848 (1983).
 - [2] A. D. Linde, Phys. Lett. B **175**, 395 (1986).
 - [3] J. García-Bellido, A. D. Linde, and D. A. Linde, Phys. Rev. D **50**, 730 (1994)
 - [4] J. García-Bellido and A. D. Linde, Phys. Rev. D **51**, 429 (1995).
 - [5] A. Starobinsky, in *Current Topics in Field Theory, Quantum Gravity and Strings* (eds. H. J. de Vega and N. Sanchez), *Lecture Notes in Physics*, Vol. 246, Springer, Heidelberg (1986), p. 107
 - [6] A. S. Goncharov, A. D. Linde, and V. F. Mukhanov, Int. J. Mod. Phys. A **2**, 561 (1987); K. Nakao, Y. Nambu, and M. Sasaki, Prog. Theor. Phys. **80**, 1041 (1988); Y. Nambu and M. Sasaki, Phys. Lett. B **219**, 240 (1989); Y. Nambu, Prog. Theor. Phys. **81**, 1037 (1989); M. Mijić, Phys. Rev. D **42**, 2469 (1990); D. S. Salopek and J. R. Bond, Phys. Rev. D **43**, 1005 (1991).
 - [7] A. D. Linde and A. Mezhlumian, Phys. Lett. B **307**, 25 (1993); A. D. Linde, D. A. Linde, and A. Mezhlumian, Phys. Rev. D **49**, 1783 (1994).
 - [8] A. Vilenkin, Phys. Rev. D **59**, 123506 (1999).
 - [9] S. Winitzki and A. Vilenkin, Phys. Rev. D **53**, 4298 (1996).
 - [10] M. Aryal and A. Vilenkin, Phys. Lett. B **199**, 351 (1987).
 - [11] A. Vilenkin, Phys. Rev. D **46**, 2355 (1992).

- [12] A. H. Guth and E. J. Weinberg, Nucl. Phys. B **212**, 321 (1983).
- [13] A. D. Linde, Phys. Lett. B **327**, 208 (1994); A. Vilenkin, Phys. Rev. Lett. **72**, 3137 (1994).
- [14] A. D. Linde and D. A. Linde, Phys. Rev. D **50**, 2456 (1994).
- [15] L. J. Garay, Int. J. Mod. Phys. A **10**, 145 (1995).
- [16] See R. Easther, B. R. Greene, W. H. Kinney, and Gary Shiu, Phys. Rev. D **64**, 103502 (2001), and references therein.
- [17] M. Lemoine, M. Lubo, J. Martin, and J.-P. Uzan, Phys. Rev. D (to appear), preprint hep-th/0109128.
- [18] J. Garriga and A. Vilenkin, Phys. Rev. D **57**, 2230 (1998).
- [19] A. Vilenkin, Phys. Rev. D **52**, 3365 (1995).
- [20] J. Feder, *Fractals*, Plenum Press, NY, 1988.
- [21] S. Winitzki and A. Vilenkin, Phys. Rev. D **61**, 084008 (2000).
- [22] R. Bousso, Phys. Rev. D **58**, 083511 (1998); Phys. Rev. D **60**, 063503 (1999).